Simplex method is not polynomial time - Klee-Minty Cube

Source: Chapter 5 of Linear and Nonlinear Programming, Luenberger and Ye

Simplex method outline: Convert problem to $A\mathbf{x} = \mathbf{b}$. Find a basic feasible solution. Perform pivot steps until no variable can increase.

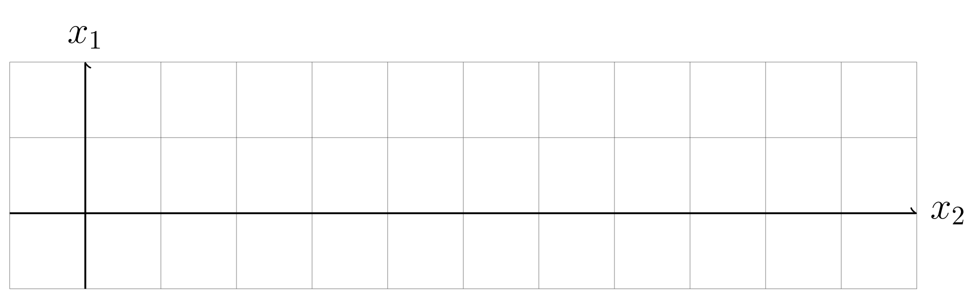
Geometry of simplex method - start at vertex of the polytope of feasible solutions and find a path on edges of the polytope to the optimal vertex.

Consider greedy way - always go in the direction that maximizes the slope in the objective function.

1: Solve the following problem using the simplex method and always pick as a pivot variable with largest coefficient in the objective function. Perhaps add slack variables.

$$(P)\begin{cases} \text{maximize} & 10x_1 + x_2 \\ \text{subject to} & \begin{array}{rcrcrl} x_1 & & & \leq & 1 & (A) \\ 20x_1 & + & x_2 & \leq & 100 & (B) \\ x_1 & & & \geq & 0 & (C) \\ & & x_2 & \geq & 0 & (D) \end{array} \end{cases}$$

2: Plot the set of feasible solution of (P) and plot basic feasible solutions. Do not include slack variables (2D).



The example from previous page generalizes to higher dimesion(s). Here is a 3D example.

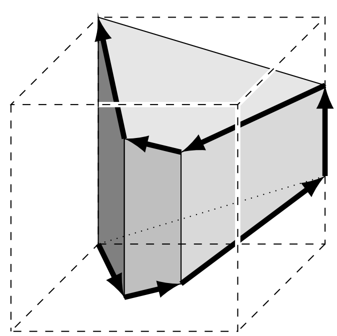
$$(P)\begin{cases} \text{maximize} & 100x_1 + 10x_2 + x_3 \\ \text{subject to} & \begin{array}{rcrcrcll} x_1 & & & & & \leq & 1 & (A) \\ 20x_1 & + & x_2 & & & \leq & 100 & (B) \\ 200x_1 & + & 20x_2 & + & x_3 & \leq & 10000 & (C) \\ x_1 & & & & & \geq & 0 & (D) \\ & & x_2 & & & \geq & 0 & (E) \\ & & & & x_3 & \geq & 0 & (F) \end{array} \end{cases}$$

Notice that a vertex is given by intersection of three of the halfspaces. That is, pick three of the equations to be satisfied with equality and it gives a vertex.

3: Steps in simplex method the solutions projected to x_1,x_2 and x_3. Fill the rest of the table.

step	x_1	x_2	x_3	value of objective	equalities
0	0	0	0	0	$(D),(E),(F)$
1	1	0	0	100	$(A),(E),(F)$
2	1	80	0	900	$(A),(B),(F)$
3	0	100	0	1000	$(D),(B),(F)$
4	0	100	8000	9000	$(D),(B),(C)$
5	1				
6					
7					

The sequence of basic feasible solutions from the simplex method corresponds to a trip in the cube like in the following sketch. Note that it is indeed just a sketch since not all coordinates are *same length.*



How many vertices will be in n dimensional cube?

4: If $n = 50$ and computer examines one million points in one second, how long will it take to finish the computation?

Klee-Minty cubes are known for different pivot rules too. But the simplex algorithm works great in practice.

The Ellipsoid Method

Problem: Let $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$. Find a point in P. (given a polytope, find one point in it)

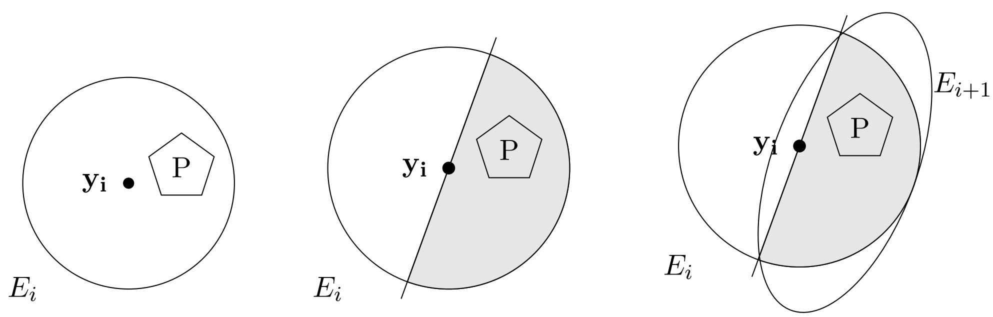
Extra assumptions:

- $\exists R \in \mathbb{R}, P \subseteq B(\mathbf{0}, R)$
- $\exists r \in \mathbb{R}, \exists \mathbf{c} \in \mathbb{R}^n, B(\mathbf{c}, r) \subseteq P$

In other words, P is in a big ball with radius R and contains a small ball of radius r.
The R and r are part of the running time.

Algorithm (can be used to solve linear programming in polynomial time. By Leonid Khachiyan):

1. $E_1 := B(0, R)$, $i := 1$
2. if center $\mathbf{y}_i$ of E_i in P, point found
3. if $\mathbf{y}_i \notin P$, there is a separating hyperplane cutting out half of E_i
4. Pick E_{i+1} to be the smallest ellipsoid containing the half of E_i that contains P
5. $i := i + 1$ and goto 2.



Claim: If $E_i \in \mathbb{R}^n$ and E_{i+1} is the smallest ellipsoid contain $\frac{1}{2}$ of E_i, then

$$\frac{volume(E_{i+1})}{volume(E_i)} < e^{\frac{-1}{2(n+1)}} < 1.$$

5: Compute an upper bound on

$$\frac{volume(E_{i+2(n+1)})}{volume(E_i)}$$

6: How many iterations of the algorithm are needed? (Use that $B(\mathbf{c}, r) \subset P$.)

One iteration takes $O(n^2)$ operations and volume of balls is at most exponential (in size of input numbers).
Not a practical algorithm in speed.